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2000 J. Phys. A: Math. Gen. 33 3669

(http://iopscience.iop.org/0305-4470/33/19/304)

J. Phys. A: Math. Gen. **33** (2000) 3669–3695. Printed in the UK PII: S0305-4470(00)06248-X

The partition algebra and the Potts model transfer matrix spectrum in high dimensions

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Received 22 July 1999

Abstract. We construct generalizations $P_n^m(Q)$ of the partition algebra $P_n(Q)$ (Martin P P 1996 *J. Algebra* **183** 319), facilitating a representation theoretic approach to the n-site transfer matrix spectrum of a high-dimensional Q-state Potts model with magnetic field and source terms (and to corresponding dichromatic polynomials). For each $Q \in \mathbb{C}$ we describe the irreducible representation theory of the sequence of algebras $P_*(Q) = \{P_n(Q) \subset P_n^1(Q) \subset P_{n+1}(Q) | \ n = 0, 1, 2, \ldots\}$ approaching the large-n limit. For each positive integer Q we extend the Potts model representation ρ_n of $P_n(Q)$ to a representation of $P_n^1(Q)$. We show how these Potts representations embed in the representation theory of the partition algebras. These results together provide a tool with which to examine the nature of physical correlation functions.

For large n the irreducible content of the Potts representations can be summarized by $\mathbb{C}S_Q \cong \mathrm{End}_{P_n(Q)}(V_Q^{\otimes n})$ and $\mathbb{C}S_{Q-1} \cong \mathrm{End}_{P_n^1(Q)}(V_Q^{\otimes n})$, where S_Q is the symmetric group, and V_Q is the space of states of a Potts spin. We show how the partition algebra formalism matches up the correlation functions of the Potts model and the corresponding absolute spectrum degeneracies of its transfer matrix.

1. Introduction

In this paper we consider the characterization by physical observables, and the degeneracies, of the transfer matrix spectrum in high-dimensional Potts models. An understanding of three-dimensional Potts models in particular would be of considerable physical interest [16, 25, 27, 45], however, despite important progress toward exact solutions to certain *special* three-dimensional models by Baxter and others (see, for example, [4,6,7,41,43,47]) relatively little is known of the spectrum in the ordinary Potts case†. By general theory the largest eigenvalue is usually non-degenerate [28, 37] and determines the free energy. Other distinct eigenvalues correspond, in theory, to distinct physical correlation functions, but it is not necessarily obvious in general which correlation function goes with which matrix eigenvalue. A physical characterization of the eigenvalues, together with their multiplicities where degenerate (n.b., degeneracies correspond to non-Abelian symmetries of the transfer matrix $\mathcal{T}$) would be a step forward.

To be precise, consider the 'minimal' multiplicities, i.e. those of eigenvalues which maintain *exact* degeneracy even if the nearest-neighbour interaction strengths are slightly different in a random way from bond to bond (or, for example, randomly bond diluted, cf [38])‡.

† In the search for exact results certain more limited analyses have proved interesting, such as at large Q [14, 46], or for finite lattices [30, 40] or, essentially equivalently, for perturbation series [18].

‡ A very important class of degeneracies occurs only at a special value of temperature/coupling constant (the vanishing of the *mass gap* [25] at a phase transition point, for example). But of course, we do not yet know the critical temperatures of Potts models above two dimensions.

For lattice models in general, the minimal multiplicities may have limited *direct* physical significance (cf [35], for example). However, in two-dimensional Potts models, where all these multiplicities are known [5,31], there is a marked difference between the minimal multiplicity behaviours of models with second- and first-order phase transitions. In the former cases the *transfer matrix algebra* [26,32,39] has only a finite number of irreducible representations, each with finite degeneracy in the thermodynamic limit; in the latter cases there are unboundedly many highly degenerate irreducibles. By analogy, data on phase structure in three and higher dimensions could also be revealed by examining degeneracies.

We will speak of directions within the transfer matrix layer as 'space' directions, and that between layers as the 'time' direction. In ordinary equilibrium statistical mechanics the distinction is, usually, for computational convenience only. In this paper we consider very high-dimensional or (equivalently on the finite lattice) spatial geometry-free models. These are, in a suitable sense, 'mean field' models—recall that spin models at four dimensions and above are expected to behave like mean field systems [2]. Mean field models are not necessarily in themselves very physical. However, they are interesting in the Potts case, since the configuration space $V_Q^{\otimes n}$ of a Potts transfer matrix layer [3] is determined only by the number of sites n, and not directly by dimension. This means that correlation functions which can be measured in any dimension can be measured in our model (excepting only that, as usual in the transfer matrix formalism, arbitrarily large separation of observables is possible only in the 'time' direction). Here the transfer matrix algebra is the image $\hat{P}_n(Q)$ of the partition algebra $P_n(Q)$ [32] on $V_Q^{\otimes n}$. Note that the natural action of the symmetric group S_Q on $V_Q^{\otimes n}$ is to apply a fixed permutation to all the Potts spin states simultaneously. This action commutes with the transfer matrix in zero magnetic field (see section 1.1), and in fact

$$\mathbb{C}S_Q \cong \mathrm{End}_{P_n(Q)}(V_Q^{\otimes n}) \qquad \text{for} \quad n > Q - 1 \tag{1}$$

(this is easy to show, for example using [24] or [36]). That is, $\mathbb{C}S_Q$ and $\hat{P}_n(Q)$ are *in Schur–Weyl duality* [44] on $V_Q^{\otimes n}$. This fact effectively determines lists of minimal spectrum multiplicities for the high-d Potts model for all $Q \in \mathbb{N}$. However, the technique is formal and it does not relate multiplicities to physical correlation functions, or address the possibility of higher multiplicities in specific transfer matrices. In this paper we index the multiplicities using the generic irreducible representation theory of the partition algebra [33], in terms of which the physical correlation functions are *intuitively* clear [32] (we will derive the connection explicitly), and many of the multiplicity bounds can be shown to be saturated.

These 'high-d' data, together with the data for two dimensions, establish a good framework for attacking the three-dimensional problem, as we will see below. The results are also interesting in their own right. They provide (1) new algebras $P_n^m(Q)$ which enable a fully algebraic transfer matrix formalism; (2) a determination of the rich exceptional structure of $P_n^1(Q)$ (using an extraordinary *Morita equivalence* [22] relating deformation algebras with different deformation parameters—something long sought in Hecke algebra representation theory for example); (3) examples of a *Jones basic construction* [15, 19] as an almost trivial application of the categorical properties of $P_n(Q)$ [34] to the Potts quotient $\hat{P}_n(Q)$; (4) a proof, using $P_n(Q)$ representation theory, that a large class of transfer matrix eigenvalues have *exactly* the representation theoretic degeneracy (i.e. that implied by equation (1)) for *any* ferromagnetic specialization of coupling parameters.

This paper is set out as follows. In section 1.1 we establish our notation through a brief summary of the relationship between transfer matrix spectra and correlation functions. In section 2 we introduce the new algebras we need. In section 3 we determine the structure of the new algebra $P_n^1(Q)$ using the known structure of $P_n(Q)$ by category theoretic means (some physicists may wish to bypass the details, but this really is the most elegant and efficient way). We put this result in the specific context of the transfer matrix algebra for the Potts model with magnetic field in section 4. In section 5 we hence determine the transfer matrix spectrum degeneracies in the generic $P_n(Q)$ framework in all cases, and use the Perron–Frobenius theorem to determine stronger bounds on these degeneracies. In section 6 we make some general remarks about a possible larger setting for, and extension of, this work. Technical results present only for their intrinsic interest are relegated to the appendixes. Thus representation theoretic results in appendix A.1 lead us, in appendix A.2, to exhibit a Jones basic construction (cf [15, 19] and references therein) on $\hat{P}_*(Q)$.

A warning is in order concerning the level of algebraic machinery employed in this paper. While every opportunity is taken to make contact with the underpinning physical model, we believe that not *all* of the nuts and bolts calculations are best conveyed in traditional physics language. And while the machinery employed is modest by modern algebraic standards, it is likely to require (of those who have not already done so) reading of at least some of the cited texts. The alternative would be a much longer paper [1]†.

1.1. On physical correlation functions

For H a graph let E_H denote its edge set and V_H its vertex set. Define graph A_l by $\mathsf{V}_{A_l} = \{1, 2, \ldots, l\}$ and $\mathsf{E}_{A_l} = \{\{i, j\} | i - j = \pm 1\}$. From [12], let $G = H \times A_l$ be given by $\mathsf{V}_G = \{(a, b) | a \in \mathsf{V}_H, b \in \mathsf{V}_{A_l}\}$ and $\{(a, b), (c, d)\} \in \mathsf{E}_G$ if either $a = c$ and $(b, d) \in \mathsf{E}_{A_l}$ or $b = d$ and $(a, c) \in \mathsf{E}_H$. Recall [3] that the Q-state Potts model is a model of classical equilibrium statistical mechanics in which the spins s_i take values from $\{1, 2, \ldots, Q\}$ and lie on the sites of a crystal lattice, which is typically a graph of the form $G = H \times A_l$. *Ab initio* H, called the *layer graph* of G, can be any graph. The Hamiltonian is

$$\mathcal{H}_{\text{Potts}} = \sum_{(i,j)\in\mathsf{E}_G} \beta_{ij} \delta_{s_i, s_j} + \sum_{i\in\mathsf{V}_G} h_i \delta_{s_i, 1}. \tag{2}$$

The homogeneous isotropic partition function [32] in zero magnetic field is usually written

$$Z_G(\beta) = \sum_{\{s\}} \exp(\mathcal{H}_{\text{Potts}}(\beta_{ij} = \beta, h_i = 0))$$

(where $\{s\}$ denotes all spin configurations) and the free energy per site is

$$F_G(\beta) = \frac{1}{l \cdot n} \ln Z_G(\beta)$$

where $n = |\mathsf{V}_H|$. For M a matrix let $\langle\langle M \rangle\rangle$ denote the sum of its entries. Then if $\mathcal{T}$ is the usual Potts transfer matrix ([3, 25], and see section 4) associated to layer graph H we can write $Z_G(\beta) = \langle\langle \mathcal{T}^l \rangle\rangle$ (we are glib about boundary conditions for brevity's sake). Thus in the thermodynamic limit of l we have $F_G(\beta) = \kappa \ln(\lambda_0(\beta))$, where κ is a constant and λ_0 is the (usually unique) largest eigenvalue of $\mathcal{T}$.

An *observable* Θ is a function of the configuration of some finite set of spins, and its *expectation value* is

$$\langle \Theta \rangle := \frac{\sum_{\{s\}} \Theta \exp(\mathcal{H}_{\text{Potts}}(\beta))}{Z_G(\beta)}.$$

Suppose we have a set of spins localized in the neighbourhood of some point $x \in \mathsf{V}_G$ (metrical notions on G are discussed in [32] section 1.2). A correlation function typically measures the rate of decrease with separation r of the degree of correlation (in the statistical mechanical sense—again see [32]) between the configuration of one such set of spins and that of another.

† This paper is in part an abbreviated version of [1], and greater detail on some points may be found there.

For example, the simplest correlation function measures the r dependence of the propensity of two spins, distance r apart, to be coaligned. In transfer matrix formalism (with β dependences implicit) correlation functions may appear in the form

$$f_{\Theta}(r) = \frac{\langle\langle \mathcal{T}^{l_1} \mathcal{O}(\Theta) \mathcal{T}^{r} \mathcal{O}(\Theta) \mathcal{T}^{l_2} \rangle\rangle}{\langle\langle \mathcal{T}^{l} \rangle\rangle} \qquad (l_1 + r + l_2 = l; l_1, l_2 \gg r \gg 1) \tag{3}$$

where $\mathcal{O}(\Theta)$ is a diagonal matrix which simply assigns the appropriate weight to each configuration of the lattice layer. Thus, just as $F \sim \ln \lambda_0$, so these functions are (for $l \gg r \gg 1$) of the form $f_{\Theta}(r) \sim \kappa' \mathrm{e}^{-r/\xi}$, where $\xi = (\ln \frac{\lambda_0}{\lambda_i})^{-1}$ is the *correlation length*, and $i \in \Gamma$, some index set for the spectrum of $\mathcal{T}$ (i.e. the spectrum is $\{\lambda_i | i \in \Gamma\}$). However, from (3), if $f_{\Theta}(r)$ is to depend non-trivially on λ_i, $\mathcal{O}(\Theta)$ must somehow suppress the larger contribution of λ_0^r in the spectrum of $\mathcal{T}^r$ (and indeed suppress all λ_j with $|\lambda_j| > |\lambda_i|$). In practice, given an arbitrary index scheme Γ, it is not obvious in general which i corresponds to which physical correlation function.

Since $\mathcal{T}$ is a representation of an element of the partition algebra $P_n(Q)$ ([32] p 211), the irreducible representation theory of $P_n(Q)$ gives a partial classification scheme for the spectrum of $\mathcal{T}$. However, there is no reason why the physical observable $\mathcal{O}(\Theta)$ should also be a representation of an element of $P_n(Q)$, so determining which irreducible representation dominates in $f_{\Theta}(r)$ for given Θ is still not necessarily trivial.

Recall ([32] ch 8, and [33]) that irreducible representations of $P_n(Q)$ are indexed by the number, m say, of 'propagating lines' and by the symmetry properties of these lines (note that this has no obvious relationship with the Q-dependent index scheme implicit in (1)—we will determine this relationship below). By reference to the strong coupling expansion formalism for Potts correlation functions [18,25] these representations will give the spectrum components corresponding to appropriately symmetrized m spin-to-m spin correlations (provided $Q \gg m$). To make this precise we can extend $P_n(Q)$ to a larger algebra $P_n^m(Q)$, constructed so that $\mathcal{O}(\Theta)$ *is* a representation of an element of the new algebra. The restriction rules for simple $P_n^m(Q)$ modules to simple $P_n(Q)$ modules can be worked out. Thus we can determine which part of the spectrum of $\mathcal{T}^r$ is picked up by $\mathcal{O}(\Theta)$ explicitly. In section 4.2 we will illustrate with concrete examples.

In the field theory sense, the new algebra $P_n^m(Q)$ allows us to incorporate *source terms* [2]. The case $P_n^1(Q)$ also allows us to build the transfer matrix for more general boundary conditions than usual, and/or a magnetic field (at the level of algebra the distinction between coupling to a frozen boundary and to a field is an entirely geometrical one).

2. Algebra definitions

Let $\mathcal{A}$ be an algebra, and $\mathcal{A}$–mod the category of finite-dimensional left $\mathcal{A}$-modules. For $M \in \mathcal{A}$–mod define $\mathrm{Ann}_{\mathcal{A}}(M) = \{a \in \mathcal{A} | aM = 0\}$. Note $\mathrm{Ann}_{\mathcal{A}}(M)$ is a double sided ideal of $\mathcal{A}$. Also define $\mathrm{Cen}_{\mathcal{A}}(M) = \mathrm{Centre}(\mathcal{A}/\mathrm{Ann}_{\mathcal{A}}(M))$, so $\mathrm{Cen}_{\mathcal{A}}(M) \subseteq \mathrm{End}_{\mathcal{A}}(M)$, and define $\mathrm{head}\ M$ as the quotient of M by the intersection of its maximal proper submodules.

For $Q \in \mathbb{N}$ let Γ_Q denote the set of *dominant weights* [17] of degree Q (also known as partitions of Q), and write $\lambda \vdash Q$ for $\lambda \in \Gamma_Q$. Let $\Gamma^n := \cup_{Q=0}^n \Gamma_Q$. For $\mu, \lambda \in \Gamma^n$ regarded as Young diagrams, write $\mu \rhd \lambda$ if $\mu - \lambda$ is a skew diagram [29] of one box.

Associate a left Young symmetrizer h'_λ to each dominant weight λ (cf the right Young symmetrizer h_λ in [11], p 250), then $\{S(\lambda) = \mathbb{C}S_Q h'_\lambda | \lambda \in \Gamma_Q\}$ is a complete set of simple left $\mathbb{C}S_Q$-modules.

2.1. Summary of the partition algebra

For M a set $\boldsymbol{E}_M$ is the set of equivalence relations on M and $\boldsymbol{S}_M$ the set of partitions of M into disjoint subsets. Recall that $\boldsymbol{E}_M$ has a natural bijection with $\boldsymbol{S}_M$. We will use them interchangeably without further comment. Note that $(\boldsymbol{E}_M, \subseteq)$ is a lattice.

Definition 1. *Let ρ be any finite equivalence relation, and $\#(\rho)$ the number of equivalence classes of ρ. For any finite sets M, N define maps*

$$\begin{aligned} &R_N : \boldsymbol{E}_{M\cup N} \to \boldsymbol{E}_{M\setminus N} \\ &R_N : \rho \mapsto (\textit{largest } \mu \in \boldsymbol{E}_{M\setminus N} \textit{ such that } \mu \subseteq \rho) \end{aligned} \tag{4}$$

and

$$\begin{aligned} &C_N : \boldsymbol{E}_{M\cup N} \to \mathbb{N}_0 \\ &C_N : \rho \mapsto \#(\rho) - \#(R_N(\rho)) \end{aligned} \tag{5}$$

and

$$\begin{aligned} &\bullet : \boldsymbol{E}_M \times \boldsymbol{E}_N \to \boldsymbol{E}_{M\cup N} \\ &(\mu, \nu) \mapsto \mu \bullet \nu \end{aligned} \tag{6}$$

where $\mu \bullet \nu$ is the smallest $\rho \in \boldsymbol{E}_{M\cup N}$ s.t. $\rho \supseteq \mu \cup \nu$ (i.e. the transitive extension of $\mu \cup \nu$).

Note that $f : M \to N$ a bijection leads naturally to a bijection $\boldsymbol{E}^f : \boldsymbol{E}_M \to \boldsymbol{E}_N$.

Let $\{\alpha, \beta, \ldots\}$ be a finite set, then define $\{\alpha, \beta, \ldots\}_{\underline{n}} = \{\alpha, \beta, \ldots\} \times \{1, 2, \ldots, n\}$, and write α_i for $(\alpha, i) \in \{\alpha, \beta, \ldots\}_{\underline{n}}$.

Let $a(\alpha, \beta)$ denote an element of $\boldsymbol{E}_{\{\alpha,\beta\}_{\underline{n}}}$. Then by $a(\delta, \gamma) \in \boldsymbol{E}_{\{\delta,\gamma\}_{\underline{n}}}$ we understand $\boldsymbol{E}^f(a(\alpha, \beta))$, where f is the isomorphism $f(\alpha_i) = \gamma_i$, $f(\beta_i) = \delta_i$ for all i. Note that $a(\alpha, \gamma) \bullet b(\gamma, \beta) \in \boldsymbol{E}_{\{\alpha,\gamma,\beta\}_{\underline{n}}}$.

For k a field we write $k\boldsymbol{E}_M$ for the free k module with basis $\boldsymbol{E}_M$.

Definition 2 (partition algebra [34]). *Let k be a field, $Q \in k$ and $n \in \mathbb{N}$. The partition algebra $P_n = kP_n(Q)$ is a k-algebra defined as follows. It has basis $\boldsymbol{E}_{\{\alpha,\beta\}_{\underline{n}}}$, i.e. it is $k\boldsymbol{E}_{\{\alpha,\beta\}_{\underline{n}}}$ as a k-space, and multiplication is given by*

$$ab = a(\alpha, \beta) * b(\alpha, \beta) = Q^{C_{\{\gamma\}_{\underline{n}}}(a(\alpha,\gamma)\bullet b(\gamma,\beta))} R_{\{\gamma\}_{\underline{n}}}(a(\alpha, \gamma) \bullet b(\gamma, \beta)). \tag{7}$$

The partition algebra product is summarized by the example in figure 1. This gives a realization of examples of partitions of $\{\alpha, \beta\}_{\underline{5}}$ as clusters (specifically, $a = \{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \{\alpha_5, \beta_1, \beta_3\}, \{\beta_2\}, \{\beta_4\}, \{\beta_5\}\}$ and $b = \{\{\alpha_1, \beta_1, \beta_2\}, \{\alpha_2, \alpha_3, \beta_3\}, \{\alpha_4, \beta_4\}, \{\alpha_5\}, \{\beta_5\}\}$) and of composition of partitions $ab = Q \cdot \{\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \{\alpha_5, \beta_1, \beta_2, \beta_3\}, \{\beta_4\}, \{\beta_5\}\}$ by an appropriate juxtaposition (cf Brauer [8] p 868).

Note that $\boldsymbol{E}^f$ with $f(\alpha_i) = \beta_i$, $f(\beta_i) = \alpha_i$ defines an isomorphism of P_n with its opposite, P_n^{op}. P_n is unital and associative, and the unit element is

$$1 = \{\{\alpha_1, \beta_1\}, \{\alpha_2, \beta_2\}, \ldots, \{\alpha_i, \beta_i\}, \ldots, \{\alpha_n, \beta_n\}\}. \tag{8}$$

Put

$$\begin{aligned} A^{i\cdot} &= \{\{\alpha_1, \beta_1\}, \{\alpha_2, \beta_2\}, \ldots \{\alpha_i\}, \{\beta_i\}, \ldots, \{\alpha_n, \beta_n\}\} && i = 1, 2, \ldots, n \\ \mathcal{I}_{ij} &= \{\{\alpha_1, \beta_1\}, \{\alpha_2, \beta_2\}, \ldots \{\alpha_i, \beta_j\}, \{\alpha_j, \beta_i\}, \ldots, \{\alpha_n, \beta_n\}\} && i, j = 1, 2, \ldots, n \\ A^{ij} &= \{\{\alpha_1, \beta_1\}, \{\alpha_2, \beta_2\}, \ldots \{\alpha_i, \beta_j, \alpha_j, \beta_i\}, \ldots, \{\alpha_n, \beta_n\}\} && i, j = 1, 2, \ldots, n. \end{aligned} \tag{9}$$

Note for $Q \neq 0$, that $e_i := \frac{A^{i\cdot}}{Q}$ and $E_i^{(n)} := \prod_{j=1}^{n-i} e_j \in P_n$ are idempotent, and [33] that

$$e_n P_n e_n \cong P_{n-1} \tag{10}$$

$$P_n / P_n e_n P_n \cong kS_n \tag{11}$$

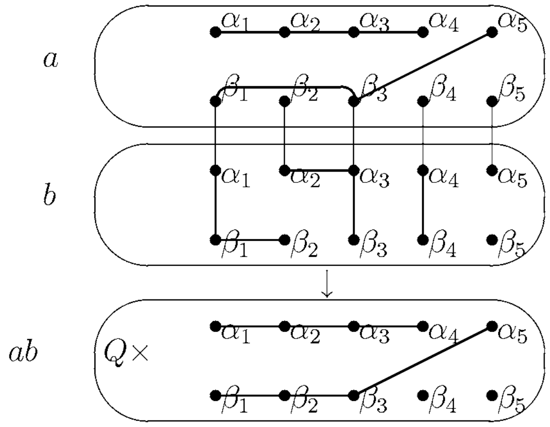

Figure 1. Pictorial realizations of partitions and their composition (we use a *vertical* stacking convention in pictures throughout).

are isomorphisms of algebras.

For $x \in \boldsymbol{E}_{\{\alpha,\beta\}_{\underline{n}}}$ define *propagating number* $\#^P(x)$ as the number of parts of x intersecting non-trivially with both $\{\alpha\}_{\underline{n}}$ and $\{\beta\}_{\underline{n}}$. Define the injection $\psi : kS_n \to P_n(Q)$ as in [34], p 322 (i.e. by $\psi(\sigma_i) = \mathcal{I}_{ii+1}$).

There is a canonical left P_n-module $P_n[i]'$ associated to any fixed propagating number i, given by exactness of the sequence of left modules

$$0 \to P_n E^{(n)}_{i-1} P_n \cap P_n E^{(n)}_i \to P_n E^{(n)}_i \to P_n[i]' \to 0. \tag{12}$$

The module $P_n[i]'$ has basis

$$B_i = \{x \in \boldsymbol{E}_{\{\alpha,\beta\}_{\underline{n}}} | \#^P(x) = i, \exists \text{ set } y \ s.t.\ x = y \cup \{\{\beta_1\}, \{\beta_2\}, \ldots, \{\beta_{n-i}\}\}\}.$$

For $k = \mathbb{C}$, the left ideal $P_n E^{(n)}_i$ is a right $\mathbb{C}S_i$-module by the action permuting $\{\beta_{n-i+1}, \ldots, \beta_n\}$. With this action, and for each $\lambda \vdash i$, put $\mathcal{S}_\lambda(n) := P_n E^{(n)}_i h'_\lambda \bmod P_n E^{(n)}_{i-1} P_n$.

The algebra injection $P_n \xrightarrow{s} P_{n+1}$ given on partitions by $s : a \mapsto a \cup \{\{\alpha_{n+1}, \beta_{n+1}\}\}$ is essentially inclusion, in the sense of $S_n \subset S_{n+1}$, and will be treated as such here. The injection s makes $P_n e_n$ a left P_n–right P_{n-1}–bimodule, so via the isomorphism in (10) there are functors

$$(P_{n-1}\text{–mod}) \xrightarrow{\mathcal{G}} (P_n\text{–mod}) \xrightarrow{\mathcal{F}} (P_{n-1}\text{–mod}) \tag{13}$$

where $\mathcal{F}(M) = e_n M$ and $\mathcal{G}(N) = P_n e_n \otimes_{P_{n-1}} N$ (respectively) are the appropriate special cases of the functors f and h of Green [17] section 6.2 (n.b., we will use the same symbols $\mathcal{F}, \mathcal{G}$ for any n). Hence, via the isomorphism in (11), we can give the following proposition.

Proposition 1. *Put* $k = \mathbb{C}$.

(i) For $Q \neq 0$, the modules $\{\mathcal{L}_\mu(n) := \mathrm{head}\, \mathcal{S}_\mu(n) | \mu \in \Gamma^n\}$ are complete set (up to isomorphism) of simple modules of $P_n(Q)$. (For $Q = 0$ this set is overcomplete.)

(ii) For $Q \notin \mathbb{N}$, $P_n(Q)$ is semi-simple (and hence each $\mathcal{S}_\mu(n)$ is simple).

(iii) For $Q \neq 0$, $P_n(Q)$ is quasi-hereditary [10] (see appendix A.3), and for each $\lambda \in \Gamma^n$, the module $\mathcal{S}_\lambda(n)$ is the standard left $P_n(Q)$ module associated to λ (i.e. the module $\Delta(\lambda)$ in [10]).

2.2. The new algebras P_n^m

Clearly, the partition algebra construction could be used to equip any $k\boldsymbol{E}_M$ with the structure of an algebra, provided $|M|$ is even. Now define

$$\boldsymbol{E}_n^m = \{a \in \boldsymbol{E}_{\{\alpha\}_{\underline{\beta, n+m}}} | \alpha_{n+j} \sim^a \beta_{n+j} \forall j = 1, 2, \ldots, m\}$$

and note that $R_{\{\beta_{n+1},\ldots,\beta_{n+m}\}} : \boldsymbol{E}_n^m \leftrightarrow \boldsymbol{E}_L$, where $L = \{\alpha\}_{\underline{n+m}} \cup \{\beta\}_{\underline{n}}$ (i.e. with $|L| = 2n + m$), is a bijection.

Proposition 2. *The subspace of $kP_{n+m}(Q)$ with basis $\boldsymbol{E}_n^m$ is a subalgebra, denoted $P_n^m = kP_n^m(Q)$.*

Note that $s(P_n) \subset P_n^1$.

Proposition 3.

(i) The algebra P_n^m is generated by the subset of P_{n+m} consisting of $1, \mathcal{I}_{ij}$ ($i, j = 1, 2, \ldots, n$), $A^{i\cdot}$ ($i = 1, 2, \ldots, n$), A^{ij} ($i, j = 1, 2, \ldots, n+m$).

(ii) For $m > 1$, and each pair i, j with $n < i < j \leqslant n+m$, A^{ij} is a central idempotent of P_n^m, and $P_n^m A^{ij} \cong P_n^{m-1}$.

From now on $k = \mathbb{C}$ unless otherwise stated. We write $P_n(Q)$ for $\mathbb{C}P_n(Q)$.

We will see in section 4 that the $A^{i\,n+1}$s allow the building of an n-site Potts transfer matrix in which spins may be coupled to a frozen boundary or, if all the $A^{i\,n+1}$s are used, a magnetic field. Thus the case $m = 1$ is sometimes called the *magnetic field algebra* (and sometimes denoted $P_{n+}(Q)$).

The left ideal $P_n^1 E_{i+1}^{(n+1)}$ is a right $\mathbb{C}S_i$ module by the action permuting $\{\beta_{n-i+1}, \ldots, \beta_n\}$. With this action, and for each $\lambda \vdash i$, put module $\mathcal{S}_\lambda^1(n) := P_n^1 E_{i+1}^{(n+1)} h'_\lambda$ mod. $P_n^1 E_i^{(n+1)} P_n^1$ (cf (12)). For example, a basis for $\mathcal{S}_{(1)}^1(1)$ is

$$\{\{\{\alpha_1\}, \{\alpha_2, \beta_2\}, \{\beta_1\}\}, \{\{\alpha_1, \alpha_2, \beta_2\}, \{\beta_1\}\}\}. \tag{14}$$

Note that

$$P_n^1 = P_n^1 E_{n+1}^{(n+1)} P_n^1 \supset P_n^1 E_n^{(n+1)} P_n^1 \supset \cdots \supset P_n^1 E_1^{(n+1)} P_n^1 \tag{15}$$

—this follows from proposition 2 of [34] on noting that $P_n^1 E_i^{(n+1)} P_n^1$ has a basis

$$B(i) = \{x \in \boldsymbol{E}_n^1 | \#^P(x) \leqslant i\}. \tag{16}$$

3. The structure of $P_n^1(Q)$

Define

$$\chi_{n+1} = \left(\prod_{i=1}^{n} (1 - A^{i\,n+1})\right).$$

Proposition 4. *Fix an integer $k \geqslant 1$ and put $\chi = \chi_{n+k}$. The composite map*

$$\chi P_n^k(Q)\chi \xrightarrow{\theta} \chi P_n^{k-1}(Q)\chi \xrightarrow{\Psi} P_n^{k-1}(Q-1) \tag{17}$$

where θ is the identity map and, with $a \in \boldsymbol{E}_{\{\alpha,\beta\}_{\underline{n}}}$, Ψ acts on $\chi a \chi$ by replacing the idempotents χ by 1, is an algebra isomorphism.

Proof. The map θ is well defined since $(\prod_{i=1}^{n}(1 - A^{i\,n+1}))A^{j\,n+1} = 0$. For Ψ we have

$$\chi A^{j\cdot} \chi \mapsto A^{j\cdot} \tag{18}$$

and

$$\Psi((\chi A^{j\cdot} \chi)^2) = \Psi((Q-1)(\chi A^{j\cdot} \chi)) = (\Psi(\chi A^{j\cdot} \chi))^2 \tag{19}$$

and so on. □

Proposition 5. *There is a Morita equivalence*

$$P_n^1(Q)\text{–mod} \sim^{ME} P_n(Q-1)\text{–mod}. \tag{20}$$

Proof. The Morita equivalence bimodules [22] are $\chi P_n^1(Q)$ and $P_n^1(Q)\chi$, where we use $\chi P_n^1(Q)\chi$ for $P_n(Q-1)$. Noting proposition 4 we need only prove that there exists a bijection

$$P_n^1(Q)\chi \bigotimes_{\chi P_n^1 \chi} \chi P_n^1(Q) \xrightarrow{\phi} P_n^1(Q). \tag{21}$$

Dlab and Ringel [13] prove that for $\mathcal{A}$ an algebra and $e \in \mathcal{A}$ an idempotent then the following are equivalent:

(i) $\mathcal{A}e\mathcal{A}$ is a projective left $\mathcal{A}$ module.
(ii) $e\mathcal{A}$ is a projective left $e\mathcal{A}e$ module, and multiplication $\mathcal{A}e \otimes_{e\mathcal{A}e} e\mathcal{A} \to \mathcal{A}e\mathcal{A}$ is bijective.

Now $P_n^1\chi$ has basis isomorphic to a subset of $\boldsymbol{E}_n^1$ with elements with β_{n+1} not connected to any other β. It follows that $P_n^1\chi P_n^1 \cong P_n^1$ as a vector space and hence as a bimodule. Thus ϕ is a bijection. □

Thus by proposition 1, we can give the following proposition.

Proposition 6.

(i) For $Q \neq 1$, the modules $\{\mathcal{L}_\mu^1(n) := \text{head}\,\mathcal{S}_\mu^1(n) \mid \mu \in \Gamma^n\}$ are a complete set (up to isomorphism) of simple modules of $P_n^1(Q)$. (For $Q = 1$ this set is overcomplete.)
(ii) For $Q \notin \mathbb{N}$, $P_n^1(Q)$ is semi-simple (and hence each $\mathcal{S}_\mu^1(n)$ is simple).
(iii) For $Q \neq 1$, $P_n^1(Q)$ is quasi-hereditary [10] (see appendix A.3), and for each $\lambda \in \Gamma^n$, the module $\mathcal{S}_\lambda^1(n)$ is the standard left $P_n^1(Q)$ module associated to λ (i.e. the module $\Delta(\lambda)$ in [10]).

The dimensions of the $\mathcal{S}_\lambda^1(n)$ modules are computed as follows.

Definition 3. *The formal infinite matrix U with row and column positions indexed by dominant weights (regarded as Young diagrams) in the standard order [29] is given by*

$$U_{\mu\nu} = \begin{cases} 1 & \mu = \nu \\ 1 & \mu = \nu + \Box \\ 0 & \text{otherwise.} \end{cases}$$

where $+\Box$ indicates adding a box, that is, $(\nu + \Box) - \nu$ is a skew diagram of one box [29].

Proposition 7. *The matrices U and $U^\dagger$ are the generic global restriction matrices for modules $\mathcal{S}_\lambda$ and $\mathcal{S}_\lambda^1$ restricted via $P_{n+1} \supset P_n^1 \supset P_n$, respectively. That is*

$$\text{Res}_{P_n^1}^{P_{n+1}} \mathcal{S}_\lambda(n+1) \cong \bigoplus_\mu (U)_{\mu\lambda} \mathcal{S}_\mu^1(n) \qquad \text{and} \qquad \text{Res}_{P_n}^{P_n^1} \mathcal{S}_\lambda^1(n) \cong \bigoplus_\mu (U^\dagger)_{\mu\lambda} \mathcal{S}_\mu(n) \tag{22}$$

(in non-semisimple cases the sums may not be direct).

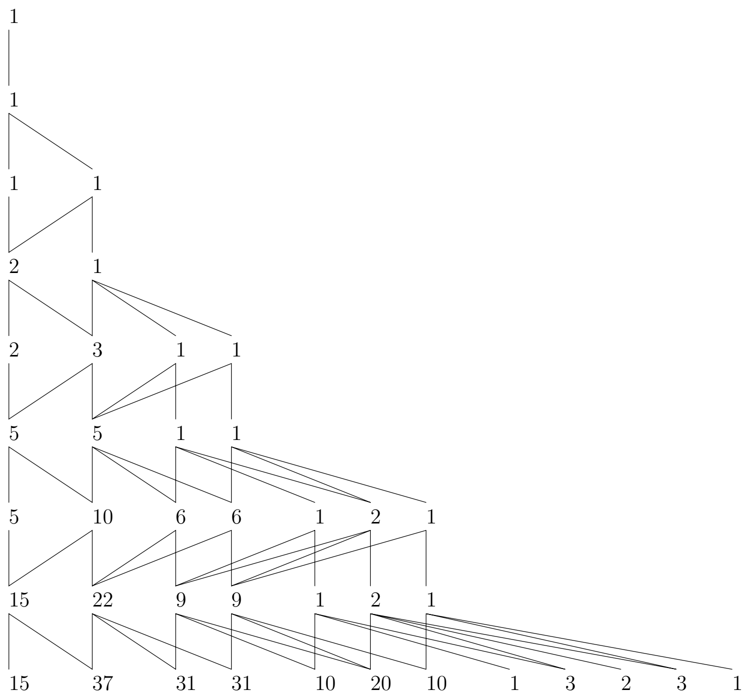

Figure 2. The 'Bratteli' diagram for the restriction of standard modules in the tower $P_* = P_0 \subset P_0^1 \subset P_1 \subset P_1^1 \subset P_2 \subset P_2^1 \subset P_3 \ldots$. The columns correspond to $\lambda = (0), (1), (2), (1^2), (3), (2,1), (1^3), \ldots$ from left to right; and each standard module $\mathcal{S}_\lambda$ (resp. $\mathcal{S}^1_\lambda$), at each level n, is represented by its dimension.

This is an obvious refinement of the case for restriction directly from P_{n+1} to P_n (see [33]).

In particular, U is lower unitriangular and begins

$$U = \begin{pmatrix} 1 & & & & & & & 0 \\ 1 & 1 & & & & & & \\ 0 & 1 & 1 & & & & & \\ 0 & 1 & 0 & 1 & & & & \\ 0 & 0 & 1 & 0 & 1 & & & \\ 0 & 0 & 1 & 1 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & \\ & & & & & & & \ldots \end{pmatrix} \qquad \text{basis:} \begin{pmatrix} (0) \\ (1) \\ (2) \\ (1^2) \\ (3) \\ (2,1) \\ (1^3) \\ \vdots \end{pmatrix}$$

and we have the generic simple module Bratteli diagram shown in figure 2.

Note that the *complete* representation theory of $P_n^1(Q)$ for each $Q \in \mathbb{C}$ is obtained by applying proposition 5 to the characterization of $P_n(Q-1)$–mod given in [34]. For example, consider the $P_3^1(Q)$ layer of the Bratteli diagram. We may determine the structure of, say, $P_3^1(2)$, which is *not* semisimple, using the $|\lambda|$-diagram results of [34] for the algebra $P_3(1)$. Reading from right to left in the layer, the modules $\mathcal{S}^1_{(1^3)}, \mathcal{S}^1_{(2,1)}, \mathcal{S}^1_{(3)}, \mathcal{S}^1_{(1^2)}$ are simple, as the corresponding $P_3(1)$-modules are; the module $\mathcal{S}^1_{(2)}$ has a maximal submodule isomorphic to $\mathcal{S}^1_{(2,1)}$, with quotient $\mathcal{L}^1_{(2)}$ of dimension $9-2=7$; $\mathcal{S}^1_{(1)}$ is simple again; and $\mathcal{S}^1_{(0)}$ has a maximal submodule isomorphic to $\mathcal{L}^1_{(2)}$, with quotient of dimension

$$\dim(\mathcal{L}^1_{(0)}) = 15 - (9-2) = 8. \tag{23}$$

We emphasize that the results obtained by this procedure completely determine the structure of the algebra.

The structure of $P^m_n(Q)$ may be determined similarly. Put $\underline{m} = \{1, 2, \ldots, m\}$ and regard $\boldsymbol{E}_{\underline{m}}$ as a partition of $\underline{m}$, so $a_i \in a$ is a subset of $\underline{m}$. For each a_i let $r_i \in \underline{m}$ denote the numerically lowest element $r_i \in a_i$. There is a decomposition of $1 \in P^m_n$ into orthogonal central idempotents given by

$$1 = \sum_{a \in \boldsymbol{E}_{\underline{m}}} \chi_n(a) \tag{24}$$

where

$$\chi_n(a) = \Bigg(\prod_{\substack{i,j \\ i \sim^a j}} A^{n+i\, n+j} \Bigg) \Bigg(\prod_{\substack{l,k \\ a_l \neq a_k}} (1 - A^{n+r_l\, n+r_k}) \Bigg).$$

(To see this expand the right-hand side of the trivial identity $1 = \prod_{\substack{i,j \\ i \neq j}} (A^{n+i\, n+j} + (1 - A^{n+i\, n+j}))$.) In the decomposition of P^m_n consequent on (24) each summand takes the form

$$P^m_n \chi_n(a) \cong P^{\#(a)}_n \prod_{\substack{l,k \in \underline{\#(a)} \\ l \neq k}} (1 - A^{n+l\, n+k})$$

using proposition 3(ii). A direct generalization of proposition 5 shows that the right-hand side is Morita equivalent to $P_n(Q')$ for a suitable choice of Q'. We thus have the following proposition.

Proposition 8. *Equivalence classes of simple modules of P^m_n are indexed by $\Gamma^n \times \boldsymbol{E}_{\underline{m}}$.*

An example of an explicit construction (in case P^2_2) is given in section 4.2.

Proposition 9. *Let $\mathcal{A}$ be an algebra and $e \in \mathcal{A}$ a primitive idempotent. Then the image of left module $\mathcal{A}e$ in a semi-simple quotient algebra of $\mathcal{A}$ is either simple or zero.*

Proof. The image of a primitive idempotent here is either primitive or zero. □

This may be applied (in general indirectly) to the modules $\mathcal{S}_\lambda$, and tells us that the elements of restriction matrices for simple modules in any tower of semi-simple quotients of $P_*(Q)$ must be no bigger than in U and $U^\dagger$.

4. The Potts model representations

For each natural number Q there is a 'Potts' representation of $P_n(Q)$

$$\rho_n : P_n(Q) \to \mathrm{End}(\otimes^n_{i=1} V^{(i)}_Q)$$

where each $V^{(i)}_Q$ is a copy of the complex vector space with basis $B = \{1, 2, \ldots, Q\}$ (i.e. the possible configurations of a single Potts spin [3]). The action of $S_n \subset P_n$ is to permute tensor factors $V^{(i)}_Q$. That is, the usual Schur–Weyl dual to the diagonal Gl_Q action [44]. The actions of $A^{i\cdot}$, A^{ij} are given, for example, in [32, p 211]—see also [24].

Let P be the $Q^2 \times Q^2$ matrix with row (column) index $i = (i_1, i_2) \in B \times B$ and

$$P_{ij} = \begin{cases} 1 & (i_1, i_2) = (j_2, j_1) \\ 0 & \text{otherwise} \end{cases}$$

and let 1_Q be the $Q \times Q$ unit matrix. Then for example we have

$$\rho_n(\mathcal{I}_{ii+1}) = 1_Q \otimes \cdots \otimes 1_Q \otimes P \otimes 1_Q \otimes \cdots \otimes 1_Q \qquad (i \leqslant n-1)$$

($n - 1$ factors, with P in the ith position so it acts on $V^{(i)}_Q \otimes V^{(i+1)}_Q$).

Proposition 10. *For each $y = (y_1, \ldots, y_m)$ with $y_j \in \{1, 2, \ldots, Q\}$, the representation ρ_n extends to a representation ρ_n^y of $P_n^m(Q)$ on putting $\rho_n^y(x) = \rho_n(x)$ as above for $x \in P_n$, but adding*

$$\rho_n^y(A^{j\,n+k}) = 1_Q \otimes \cdots \otimes 1_Q \otimes D_{y_k} \otimes 1_Q \otimes \cdots \otimes 1_Q \qquad (j \leqslant n)$$

where $D_l \in \mathrm{End}(V_Q)$ is in the jth position and is given by

$$(D_l)_{ab} = \begin{cases} 1 & \text{if } a = b = l \\ 0 & \text{otherwise} \end{cases}$$

and

$$\rho_n^y(A^{n+k_1\, n+k_2}) = \begin{cases} \rho_n(0) & y_{k_1} \neq y_{k_2} \\ \rho_n(1) & y_{k_1} = y_{k_2}. \end{cases}$$

The orbits of the natural diagonal S_Q action on y produce isomorphic representations.

Definition 4. *For given n, Q*

$$\hat{P}_n(Q) = P_n(Q)/\mathrm{Ann}_{P_n(Q)}(V_Q^{\otimes n}) \qquad \text{and} \qquad \hat{P}_n^1(Q) = P_n^1(Q)/\mathrm{Ann}_{P_n^1(Q)}(V_Q^{\otimes n}).$$

We will use $\hat{P}$ or $\hat{P}_n$ for $\hat{P}_n(Q)$ where no ambiguity arises, and $\hat{P}_n^1$ similarly.

For v_i, v_{ij}, h_i complex scalars let $\mathcal{X} \in P_n^1$ be

$$\mathcal{X} = \left(\prod_{i \in \mathsf{V}_H} (v_i 1 + A^{i\cdot}) \right) \left(\prod_{(i,j) \in \mathsf{E}_H} (1 + v_{ij} A^{ij}) \right) \prod_{i \in \mathsf{V}_H} (1 + (\mathrm{e}^{h_i} - 1) A^{i\,n+1}).$$

Then the Potts model transfer matrix may be written in the form

$$\mathcal{T} = \rho_n^y(\mathcal{X}) \tag{25}$$

where $v_i = \mathrm{e}^{\beta_i} - 1$, $v_{ij} = \mathrm{e}^{\beta_{ij}} - 1$. Here β_x is the Potts coupling strength on an edge corresponding to index x and h_i is the strength of Potts coupling of site i to a magnetic field pointing in direction y_1. Note that any hypercubical lattice transfer matrix can be built by appropriate choice of graph H. Similarly, we may construct a 'universal' quantum spin Hamiltonian with parameter d:

$$\mathcal{H} = \sum_{i=1}^{n} A^{i\cdot} + \frac{1}{d} \sum_{\langle i,j \rangle \in \mathsf{E}_H} A^{ij}$$

whose spectrum contains that of all concrete Hamiltonians obtained by mapping into a particular representation (that is, just as the appropriate image of

$$\mathcal{H}_{d=1} = \sum_{i=1}^{n} A^{i\cdot} + \sum_{i=1}^{n-1} A^{i\,i+1}$$

gives the usual one-dimensional $U_q sl_2$ invariant XXZ quantum spin chains).

4.1. The spectrum of $\mathcal{X}$ and the transfer matrix $\mathcal{T}$

We now assume H is the complete graph. Note that $\mathcal{T}$ is a positive matrix for real β_x, and hence trivially a *primitive* matrix (matrix notations for this section are taken from Seneta [42, section 1.1]). Recall that any primitive matrix has a unique largest magnitude eigenvalue, which is positive, by the Perron–Frobenius theorem. In fact all the eigenvalues of $\mathcal{T}$ are clearly real and non-negative, but the other multiplicities are not obvious.

On the other hand, the spectrum of $\mathcal{T}$ is a subset (up to multiplicities) of the spectrum of $\mathcal{X}$, and the spectrum of $\mathcal{X}$ is the union of the spectra of $\mathcal{X}$ acting on each generically simple module of P_n (or P_n^1). It is readily verified that the multiplicity of $\mathcal{S}_\lambda(n)$ in $P_n[i]'$ is $\dim(S(\lambda))$.

Proposition 11. *For each i, $\mathcal{X}(\beta_x \in \mathbb{R}_+, h_i = 0)$ acts as a primitive matrix on B_i.*

Proof. It is proved in [33] that for $i \neq n$ for all $x_1, x_2 \in B_i$, there exists $z \in (P_n A^{1\cdot} P_n \cup \{1\}) \cap \boldsymbol{E}_n^0$ such that $x_1 = z x_2$. Further, every such z induces a linear transformation on $x \in B_i$ with non-negative coefficients. Since every z appears in some power of $\mathcal{X}$ with a positive coefficient in the given parameter regime we are done. □

Since there are at most two simples with multiplicity 1 in $P_n[i]'$ it follows that the largest eigenvalue is associated to one of these. Since the corresponding eigenvector can be taken positive we may deduce from the form of the Young symmetrizer the following.

Proposition 12. *For each i, the unique largest eigenvalue of $\mathcal{X}(\beta_x \in \mathbb{R}_+, h_i = 0)$ on $P_n[i]'$ is associated to the submodule $\mathcal{S}_{(i)}$.*

Now suppose that the corresponding eigenvector v lies in $I_{(i)}$, a proper submodule of $\mathcal{S}_{(i)}$. Then by elementary category theory arguments $E_m^{(n)} v = 0$ for sufficiently small $m \geqslant i$, a contradiction of the positivity of coefficients in v. Thus, we have the following proposition.

Proposition 13. *The unique largest eigenvalue of $\mathcal{X}(\beta_x \in \mathbb{R}_+, h_i = 0)$ on $P_n[i]'$ is associated to the unique simple quotient of $\mathcal{S}_{(i)}$.*

We will shortly establish when this simple module appears in ρ_n.

Note that by *varying* the v_x and h_i parameters in the argument of the right-hand side of (25) such $\mathcal{X}$ operators can be made to generate all the $A^{i\cdot}$ and A^{ij} operators in P_n or P_n^1. With the parameters regarded as indeterminate the multiplicities of irreducible representations in the representation ρ_n thus determine the minimal multiplicities of the transfer matrix spectrum. In other words, to determine the multiplicities we make the decomposition of ρ_n into simple $P_n(Q)$ modules $\mathcal{L}_\mu$ (resp. ρ_n^y into simple $P_n^1(Q)$ modules $\mathcal{L}_\mu^1$—n.b. y is a one-tuple here):

$$\rho_n \cong \bigoplus_{\mu \in \Gamma} M_\mu^Q \mathcal{L}_\mu \qquad \rho_n^y \cong \bigoplus_{\mu \in \Gamma'} L_\mu^Q \mathcal{L}_\mu^1 \tag{26}$$

where the index set Γ has to be understood in terms of correlation functions, i.e. as a subset of Γ_n. A version of the first of these decompositions was computed by Jones [24] with Γ_Q as index set. Unfortunately, this gives no match to correlation functions, but we will derive the injection $J : \Gamma_Q \hookrightarrow \Gamma^n$ which makes this match.

In the sense that high-dimensional lattices have high coordination number (and since our layer graph is the complete graph [11] it has high coordination number) we may think of this as a high-dimensional Potts model. Our eventual objective is to deal with (2 + 1)-dimensional models, but this high-dimensional case is a convenient and illuminating first step.

Recall [3] that $Z(v = -1)$ is the colouring polynomial of the corresponding graph (the Potts model 'lattice'). For example,

$$\operatorname{tr}\left(\prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (\rho_n(1) - \rho_n(A^{ij}))\right) = 0 \qquad \text{if} \quad n > Q \tag{27}$$

since this trace is the number of ways of colouring the nodes of the n node complete graph with Q colours, such that each node is coloured differently. Since all the factors in the trace's argument are diagonal matrices with each entry either 1 or 0 we deduce here that

$$\prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (\rho_n(1) - \rho_n(A^{ij})) = 0 \qquad \text{if} \quad n > Q. \tag{28}$$

This will be useful shortly.

4.2. Potts correlation functions

Let us consider the case of zero magnetic field. In terms of equation (3) the one spin-to-one spin correlation has a layer operator of the form

$$\mathcal{O}_{(1\ \text{spin})} = \begin{pmatrix} 1-1/Q & 0 & \cdots & 0 \\ 0 & -1/Q & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & -1/Q \end{pmatrix} \otimes 1_Q \otimes \cdots \otimes 1_Q \tag{29}$$

(this is chosen so that $\langle\langle \mathcal{T}^{l_1}\mathcal{O}_{(1\ \text{spin})}\mathcal{T}^{l-l_1}\rangle\rangle = 0$, removing the need for *subtracted* correlations [25]). Thus in the Potts representation $\rho_n^1(\sigma_{(1)}) = \mathcal{O}_{(1\ \text{spin})}$, where $\sigma_{(1)} := A^{1\,n+1} - \frac{1}{Q}$. We want to determine which part of the transfer matrix spectrum contributes to the corresponding correlation function (again, in the sense of (3)). This task is complicated by the fact that the operator is not in the zero-field transfer matrix algebra $P_n(Q)$.

We work by passing to the generically simple '1 propagating line' representation of P_n^1 (i.e. that with basis elements with $\#^P(x) = 1$ [32]). For example, let us write the basis for $\mathcal{S}^1_{(1)}(1)$ given in (14) as $\{((1)(2)'), ((12)')\}$ (here $((1)(2)') = \{\{\alpha_1\}, \{\alpha_2, \beta_2\}\{\beta_1\}\}$—shorthand notation from [33, section 6.2]). A basis explicitly manifesting the restriction

$$\text{Res}^{P_n^1}_{P_n}(\mathcal{S}^1_{(1)}(1)) = \mathcal{S}_{(0)}(1) \oplus \mathcal{S}_{(1)}(1)$$

(i.e. in which the transfer matrix is diagonal) is then $\{((1)(2)'), ((12)') - \frac{1}{Q}((1)(2)')\}$. In this basis the representation of the observable is

$$R\left(A^{1\,n+1} - \frac{1}{Q}\right) = \begin{pmatrix} 0 & 1 \\ \frac{Q-1}{Q^2} & \frac{Q-2}{Q} \end{pmatrix}$$

thus in this basis

$$\frac{\mathcal{T}^{l_1}\mathcal{O}\mathcal{T}^r\mathcal{O}\mathcal{T}^{l_2}}{\langle\langle \mathcal{T}^l\rangle\rangle} \sim \frac{1}{\lambda_0^l}\begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix}^{l_1} \begin{pmatrix} 0 & 1 \\ \frac{Q-1}{Q^2} & \frac{Q-2}{Q} \end{pmatrix} \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix}^{r} \begin{pmatrix} 0 & 1 \\ \frac{Q-1}{Q^2} & \frac{Q-2}{Q} \end{pmatrix} \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix}^{l_2}$$

$$= \begin{pmatrix} (\frac{\lambda_1}{\lambda_0})^r \frac{Q-1}{Q^2} & (\frac{\lambda_1}{\lambda_0})^{r+l_2} \frac{Q-2}{Q} \\ (\frac{\lambda_1}{\lambda_0})^{r+l_1} \frac{Q-1}{Q^2}\frac{Q-2}{Q} & (\frac{\lambda_1}{\lambda_0})^{l_1+l_2} \frac{Q-1}{Q^2} + (\frac{\lambda_1}{\lambda_0})^l \frac{(Q-2)^2}{Q^2} \end{pmatrix}$$

so the long-range correlation function is dominated by $(\frac{\lambda_1}{\lambda_0})^r$ as we would hope.

Consideration of the dispersion relations not withstanding, this is a workable paradigm for all correlation functions. Suppose we wish to see correlations depending on λ_i rather than λ_1. If $R(\mathcal{T})$ is diagonal with λ_0 in the first position and λ_i in the ith position, then it is sufficient for $R(\mathcal{O})$ to take any form in which its first row is zero except in the ith position.

One further example will establish the pattern. Let a, b be elements of the defining basis of P_n^m, and c the partition part of ab. Let $\#^m(a)$ denote the number of propagating lines (parts contributing to $\#^P(a)$) involving the nodes numbered $n+1, n+2, \ldots, n+m$ (we will call these the *higher* nodes). Then $\#^m(c) \leqslant \#^m(a), \#^m(b)$, and similarly for $(\#^P - \#^m)(c)$. Therefore, ideals of P_n^m are filtered by both numbers. That is to say, there is an ideal filtration whose sections each have basis characterized by a fixed value of $\#^m$ (between 1 and m) and of $(\#^P - \#^m)$ (between 0 and n).

Thus in particular there is a generically simple module of P_2^2 with two propagating lines both involving the higher nodes, called $\mathcal{S}^2_{(0;2)}(2)$, with ordered basis

$$\epsilon = (((1)(2)(3)'(4)'), ((12)(3)'(4)'), ((2)(13)'(4)'), ((1)(23)'(4)'), ((123)'(4)'),$$
$$((2)(3)'(14)'), ((1)(3)'(24)'), ((3)'(124)'), ((13)'(24)'), ((23)'(14)')).$$

A basis manifesting the decomposition

$$\mathrm{Res}^{P_2^2}_{P_2}(\mathcal{S}^2_{(0;2)}(2)) = \mathcal{S}_{(0)}(2) \oplus 2.\mathcal{S}_{(1)}(2) \oplus \mathcal{S}_{(2)}(2) \oplus \mathcal{S}_{(1^2)}(2)$$

is

$$\left(\epsilon_1, \epsilon_2, \epsilon_3 - \frac{1}{Q}\epsilon_1, \epsilon_4 - \frac{1}{Q}\epsilon_1, \epsilon_5 - \frac{1}{Q}\epsilon_2, \epsilon_6 - \frac{1}{Q}\epsilon_1, \epsilon_7 - \frac{1}{Q}\epsilon_1, \epsilon_8 - \frac{1}{Q}\epsilon_2, x, x'\right)$$

where

$$x = \left(1, -1, \frac{1-Q}{2}, \frac{1-Q}{2}, Q-1, \frac{1-Q}{2}, \frac{1-Q}{2}, Q-1, \frac{(Q-2)(Q-1)}{2}, \frac{(Q-2)(Q-1)}{2}\right) \cdot \epsilon^t$$

(t denotes transpose) and

$$x' = (0, 0, 1, -1, 0, 1, -1, 0, Q, -Q) \cdot \epsilon^t.$$

Hence

$$\sigma_{(2)} = (1 - A^{12}) \left(\left(A^{1\,n+1} - \frac{1}{Q-2}\right)\left(A^{2\,n+2} - \frac{1}{Q-2}\right) + \left(A^{1\,n+2} - \frac{1}{Q-2}\right)\left(A^{2\,n+1} - \frac{1}{Q-2}\right) - \frac{2}{(Q-1)(Q-2)^2}\right)$$

and

$$\sigma_{(1^2)} = \left(A^{1\,n+1} - \frac{1}{Q}\right)\left(A^{2\,n+2} - \frac{1}{Q}\right) - \left(A^{1\,n+2} - \frac{1}{Q}\right)\left(A^{2\,n+1} - \frac{1}{Q}\right)$$

are elements of the algebra with the required properties. On passing to the Potts representation $\rho^{(1,2)}$ we find, for example with $Q = 5$,

$$\mathcal{O}_{(2)} \text{ diagonal with entries: } \begin{cases} 0 & s_1 = s_2 \\ \frac{1}{2} & s_1 = 1,\ s_2 = 2 \text{ or } s_1 = 2,\ s_2 = 1 \\ -\frac{1}{6} & s_1 \in \{1, 2\},\ s_2 \notin \{1, 2\} \\ -\frac{1}{6} & s_2 \in \{1, 2\},\ s_1 \notin \{1, 2\} \\ \frac{1}{6} & s_1 \neq s_2,\ s_1, s_2 \notin \{1, 2\} \end{cases}$$

$$\mathcal{O}_{(1^2)} \text{ diagonal with entries: } \begin{cases} 0 & s_1 = s_2 \\ \frac{3}{5} & s_1 = 1,\ s_2 = 2 \\ -\frac{3}{5} & s_1 = 2,\ s_2 = 1 \\ \frac{1}{5} & s_1 = 2,\ s_2 \notin \{1, 2\} \text{ or } s_2 = 1,\ s_1 \notin \{1, 2\} \\ -\frac{1}{5} & s_1 = 1,\ s_2 \notin \{1, 2\} \text{ or } s_2 = 2,\ s_1 \notin \{1, 2\} \\ 0 & \text{otherwise.} \end{cases}$$

Note, neither of these is defined unless $Q > 2$, and $\mathcal{O}_{(2)}$ only if $Q > 3$.

These operators are somewhat complicated compared to (29), however, our construction shows that they lead to two *different* types of correlation function on the states of pairs of spins. Note that this result is consistent with traditional strong coupling formalism (cf [25]), since above two dimensions there are two ways of connecting a pair of spins to a pair of spins. The generalization is straightforward. For Q sufficiently large (see section 5), we have, for $\lambda \in \Gamma^n$, an observable layer operator $\mathcal{O}_\lambda$ with correlation function determined by the largest eigenvalue in the λ sector. It is a correlation on $|\lambda|$ to $|\lambda|$ spins, with their symmetries under permutation within the layer determined by λ. It remains now to determine which of these correlation functions survive at a given Q.

4.3. Potts configuration space $V_Q^{\otimes n}$ as an $S_Q \otimes P_n(Q)$ module

Define orthogonal matrices

$$S = \begin{pmatrix} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & \ddots & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T = \begin{pmatrix} 0 & 1 & & & & & & \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & \ddots & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \in \mathrm{End}(V_Q)$$

(blank entries 0) cf [32, ch 10]. Then using symmetric group cycle notation

$$\rho_n'((12)) = \otimes_{i=1}^n S \qquad \text{and} \qquad \rho_n'((12\ldots Q)) = \otimes_{i=1}^n T \tag{30}$$

generate a representation of the symmetric group S_Q on $V_Q^{\otimes n}$. It is easy to check that these matrices commute with the representation of $P_n(Q)$ above, and that the $S_{Q-1} \subset S_Q$ which fixes y commutes with $P_n^1(Q)$. In fact,

$$\mathbb{C}S_Q \cong \mathrm{End}_{P_n(Q)}(V_Q^{\otimes n}) \qquad \text{and} \qquad \mathbb{C}S_{Q-1} \cong \mathrm{End}_{P_n^1(Q)}(V_Q^{\otimes n}). \tag{31}$$

The first of these is proved in [24], and the second requires a straightforward generalization of this proof. Thus ρ_n (resp. ρ_n^y) is extended by (30) to a representation of $S_Q \otimes P_n(Q)$ (resp. $S_{Q-1} \otimes P_n^1(Q)$).

Note that the multiplicities of irreducible representations of S_Q (or $S_{Q-1} \subset S_Q$) in ρ_n' can be worked out using the Clebsch–Gordan (Gamba and Radicati) rules [20, 23, 24]. Regard k as the trivial kS_{Q-1}-module. Then as a kS_Q-module $V_Q = \mathrm{Ind}_{kS_{Q-1}}^{kS_Q} k$ (i.e. the permutation module $P((Q-1, 1))$). As a representation of S_Q we can thus decompose ρ_1' as a sum of irreducibles:

$$\rho_1' \cong S((Q)) \oplus S((Q-1,1)) \cong S\left(\square\square\square\square\,\boxed{\cdots\cdots}\,\square\right) \oplus S\left(\begin{array}{l}\square\square\square\,\boxed{\cdots\cdots}\,\square \\ \square\end{array}\right)$$

(readily verified by taking traces). Then

$$\rho_n' \cong \rho_{n-1}' \otimes \rho_1' \qquad \text{and} \qquad V_Q^{\otimes n} = \mathrm{Ind}_{kS_{Q-1}}^{kS_Q} V_Q^{\otimes n-1}. \tag{32}$$

We may summarize the effect of direct producting any irreducible representation $S(\mu)$ with ρ_1' as

$$S(\mu) \otimes \rho_1' \cong \mathrm{Ind}_{kS_{Q-1}}^{kS_Q}(\mathrm{Res}_{kS_{Q-1}}^{kS_Q} S(\mu)) \cong \bigoplus_{\nu\in\Gamma_Q} U_{\gamma,\nu}^{(Q)} \bigoplus_{\gamma\in\Gamma_{Q-1}} U_{\gamma,\mu}^{(Q)} S(\nu) \cong \bigoplus_{\nu\in\Gamma_Q} N_{\mu,\nu}^{(Q)} S(\nu) \tag{33}$$

where $U_{\gamma,\mu}^{(Q)} = 1$ if the Young diagram of μ is obtained from that of γ by adding a box, and is zero otherwise; and $N^{(Q)} = (U^{(Q)})^t U^{(Q)}$. We may then apply this iteratively to (32).

For example, with $Q = 4$ the muliplicities of irreducibles in ρ_n' are given by the entries of the column vector

$$\left(\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \right)^{n-1} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= (N^{(4)})^{n-1} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

indexed by $\Gamma_4 = \{(4), (3, 1), (2, 2), (2, 1, 1), (1^4)\}$ (in that order). These multiplicities will be the dimensions of the irreducible representations of the centralizer algebra $\mathrm{End}_{\mathbb{C}S_Q}(V_Q^{\otimes n})$, which is $\hat{P}_n$ by the duality of (31). For $Q = 4$ these vectors begin, with $n = 1, 2, 3, 4, 5, \ldots$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 10 \\ 5 \\ 6 \\ 1 \end{pmatrix}, \begin{pmatrix} 15 \\ 36 \\ 21 \\ 28 \\ 7 \end{pmatrix}, \begin{pmatrix} 51 \\ 136 \\ 85 \\ 120 \\ 35 \end{pmatrix}, \ldots. \tag{34}$$

Note, it is a simple matter so to check (23) (although note that *none* of the other simple dimensions in $P_3^1(2)$ can be checked in this way!).

5. Irreducible content of the Potts representations

Now we determine the irreducible content of ρ_n *in terms of correlation functions* (i.e. in terms of the index set Γ^n).

Definition 5. *With $|\lambda|$-diagrams defined as in [34]:*

$$\Gamma_Q^n = \{\lambda \in \Gamma^n \mid |\lambda|\text{-diagram of } \lambda \text{ contains no } \lambda_{ij} \geqslant Q\}.$$

Note (from proposition 9 of [34]) that Γ_Q^n is in general a proper subset of an index set for connected components of the indecomposable projective modules quiver diagram of $P_n(Q)$ (cf [32] ch 8).

Proposition 14. *For $n > Q$ there is a bijection $\mathcal{I} : \Gamma_Q^n \to \Gamma_Q$ given by*

$$\mathcal{I} : \lambda \mapsto (Q - |\lambda|, \lambda_1, \lambda_2, \ldots).$$

Proof. The definition of Γ_Q^n ensures that $\lambda_1 \leqslant Q - |\lambda|$, and the inverse map simply deletes the first row. $\square$

For example

$$\Gamma_4^n = \{(0), (1), (2), (1^2), (1^3)\} \mapsto \{(4), (3, 1), (2, 2), (2, 1^2), (1^4)\} = \Gamma_4$$

(in that order).

Let $\mathcal{P}$ be the induction functor from P_n–mod, to $\hat{P}_n$–mod (any n). Thus $\mathcal{P}(\rho_n) = \rho_n$.

Proposition 15. *For $n > Q$ the Potts representation obeys*

$$\mathcal{F}(\rho_n) \cong \rho_{n-1} \tag{35}$$

and

$$\mathcal{PG}(\rho_{n-1}) \cong \mathcal{G}(\rho_{n-1})/\mathrm{Ann}_{P_n}(V_Q^{\otimes n}) \cong \rho_n. \tag{36}$$

Proof. From (28) we see that $\rho_n(1) \in \rho_n(P_n A^{ij} P_n)$ for $n > Q$ (any i, j). But $A^{ij} = A^{ij} A^{i\cdot} A^{ij}$ so $\rho_n(1) \in \rho_n(P_n e_n P_n)$ for $n > Q$. It follows that

$$\rho_n(P_n) = \rho_n(P_n e_n P_n) \qquad (n > Q). \tag{37}$$

Considering (37), $\mathcal{F}$ and $\mathcal{G}$ must induce equivalence functors between $\hat{P}_n$–mod and $\hat{P}_{n-1}$–mod. In particular, comparing the definition $\mathcal{F} : M \mapsto A^{n\cdot} M$ with (48) verifies (35). □

Note by proposition 15 that the subset of Γ_n indexing irreducible representations of each successive $\hat{P}_n$ after $n = Q$ is the same, and that the multiplicity of each irreducible representation so indexed is independent of n (once n is large enough for it to be defined)—cf [24].

For each $\lambda \vdash n$ there is a unique non-zero $\mathbb{C}S_n$ idempotent $Y_\lambda \in \mathbb{C}h'_\lambda h_\lambda$. We define

$$e_\lambda = \psi(Y_\lambda) \in P_n(Q). \tag{38}$$

Recall that $\mathbb{C}S_n Y_\lambda \cong \mathrm{S}(\lambda)$ [23], and that there exists a primitive idempotent I_λ in $P_n(Q)$ such that

$$I_\lambda \equiv e_\lambda \qquad \mathrm{mod}\ P_n A^{n\cdot} P_n \tag{39}$$

and $P_n(Q) I_\lambda \twoheadrightarrow \mathcal{S}_\lambda(n)$ [33]. Thus $I_\lambda \mathcal{S}_\lambda(n) \neq 0$.

Proposition 16. *For $Q < 2n$*

$$\rho_n(e_{(n)}) \in \rho_n(P_n A^{n\cdot} P_n).$$

Proof. Let K_{2n} be the $2n$ node complete graph and consider any bond subgraph G of this with the nodes arranged as two layers, each of n nodes. The *colouring matrix* $C(G)$ for this system at fixed Q is the element of $\mathrm{End}(V_Q^{\otimes n})$ with i, jth entry 1 if spin configuration (i, j) represents a colouring of G, and 0 otherwise (so $\sum_{ij} C_{ij}(G)$ is the evaluation of the colouring polynomial at Q and $C(K_{2n}) = 0$ for $Q < 2n$). This matrix cannot in general be written as a product of transfer matrices of the form $\mathcal{T}$, but it can be written as a linear combination of transfer matrices each modified by a permutation of the top layer of nodes, using the bond removal algorithm for the dichromatic polynomial $D(G)$ for graph G.

Recall that if $G - \{i, j\}$ denotes G with bond $\{i, j\}$ removed and $G/\{i, j\}$ denotes G with $\{i, j\}$ removed and vertices i, j identified, then

$$D(G) = D(G - \{i, j\}) + vD(G/\{i, j\}). \tag{40}$$

In our case $v = -1$ and this extends by linearity to

$$C(G) = C(G - \{i, j\}) + vC(G/\{i, j\}). \tag{41}$$

Starting from $G = K_{2n}$, it requires the application of the removal algorithm to $n(n-1)/2$ of the n^2 bonds of form $\{\alpha_i, \beta_j\}$ (α_i in the top layer, β_j in the bottom) to get a right-hand side in which each term is a product of transfer matrices. To see this consider

$$C\left(G-\sum_{i=1}^{n-1}\sum_{j=1}^{n-i}\{\alpha_i,\beta_j\}\right)=\rho_n\left(\left(\prod_{i\neq j=1}^{n}(1-A^{ij})\right)(-1+A^{n\cdot})\left(\prod_{j=1}^{n-1}(1-A^{nj})\right)\right.$$
$$\times(-1+A^{n-1\cdot})\left(\prod_{j=1}^{n-2}(1-A^{n-1j})\right)(-1+A^{n-2\cdot})$$
$$\left.\times\left(\prod_{j=1}^{n-3}(1-A^{n-2j})\right)\cdots(-1+A^{1\cdot})\left(\prod_{i\neq j=1}^{n}(1-A^{ij})\right)\right). \tag{42}$$

Each of the graphs produced has at least n connections from top to bottom and hence the argument of $\rho_n(-)$ for each one is non-vanishing mod $P_nA^{n\cdot}P_n$ (equation (42) in particular has argument congruent to ± 1 mod $P_nA^{n\cdot}P_n$). On the other hand, the original $G = K_{2n}$ is invariant under the action of $\mathrm{S}(n)$ which permutes the top nodes, so $C(K_{2n})$ has non-vanishing coefficient of $\rho_n(e_{(n)})$. $\square$

For an example, consider figure 3. There, taking care of signs

$$A = B - C = (D - F) - E = ((G - H) - (J - L)) - (I - K)$$

and collecting the leading terms in each of the last six diagrams respectively

$$\begin{aligned} A &= (((-\mathcal{I}_{23}+\cdots) - (\mathcal{I}_{(132)}+\cdots)) - ((\mathcal{I}_{(123)}+\cdots) \\ &\quad -(-\mathcal{I}_{13}+\cdots))) - ((\mathcal{I}_{23}+\cdots) - (-1+\cdots)) \\ &= -e_{(3)}+\cdots \end{aligned}$$

where $\mathcal{I}_{(123)}$ denotes the image of the cycle (123) in $P_3(Q)$, and $\cdots$ denotes elements of $P_nA^{n\cdot}P_n$.

By similar arguments we arrive at the more general result given in the following proposition.

Proposition 17. *For* $\lambda \vdash n$, $\lambda \notin \Gamma^n_Q$ *then*

$$\rho_n(e_\lambda) \in \rho_n(P_nA^{n\cdot}P_n).$$

A corollary is the following proposition.

Proposition 18. *Let* $\lambda \in \Gamma^n$, *then*

(1) $M^Q_\lambda = 0$ *if and only if* $\lambda \notin \Gamma^n_Q$.

(2) $L^Q_\lambda = 0$ *if and only if* $\lambda \notin \Gamma^n_{Q-1}$.

Proof. ('If' parts): (1) consider $\lambda \vdash n$. Then $\mathcal{S}_\lambda = \mathcal{L}_\lambda$, $A^{n\cdot}\mathcal{L}_\lambda = 0$, and $I_\lambda\mathcal{L}_\lambda \neq 0$, thus by equation (39) $e_\lambda\mathcal{L}_\lambda \neq 0$. Proposition 17 implies that

$$e_\lambda V_Q^{\otimes n} \subseteq P_nA^{n\cdot}V_Q^{\otimes n} \qquad (\text{for} \quad \lambda \notin \Gamma^n_Q).$$

Thus $P_ne_\lambda V_Q^{\otimes n} \subseteq P_nA^{n\cdot}V_Q^{\otimes n}$ for $\lambda \notin \Gamma^n_Q$. Now suppose $\mathcal{L}_\lambda \hookrightarrow V_Q^{\otimes n}$, then $\mathcal{L}_\lambda \hookrightarrow P_ne_\lambda V_Q^{\otimes n}$ (since $\mathcal{L}_\lambda$ simple), but $\mathcal{L}_\lambda \not\hookrightarrow P_nA^{n\cdot}V_Q^{\otimes n}$ giving a contradiction when $\lambda \notin \Gamma^n_Q$. Thus $\mathcal{L}_\lambda \not\hookrightarrow V_Q^{\otimes n}$ for $\lambda \notin \Gamma^n_Q$, and by proposition 15 $\mathcal{L}_\lambda \not\hookrightarrow V_Q^{\otimes m}$ for all $m \geqslant n$.

(2) Similar, but one colour fixed in the P^1_n version of proposition 17.

So far we established the 'if' parts. This tells us some of the rows and columns which are lost in truncating U and $U^\dagger$ when moving to the quotient (cf proposition 9). Suppose this is *all* that is lost, then we would have the following proposition.

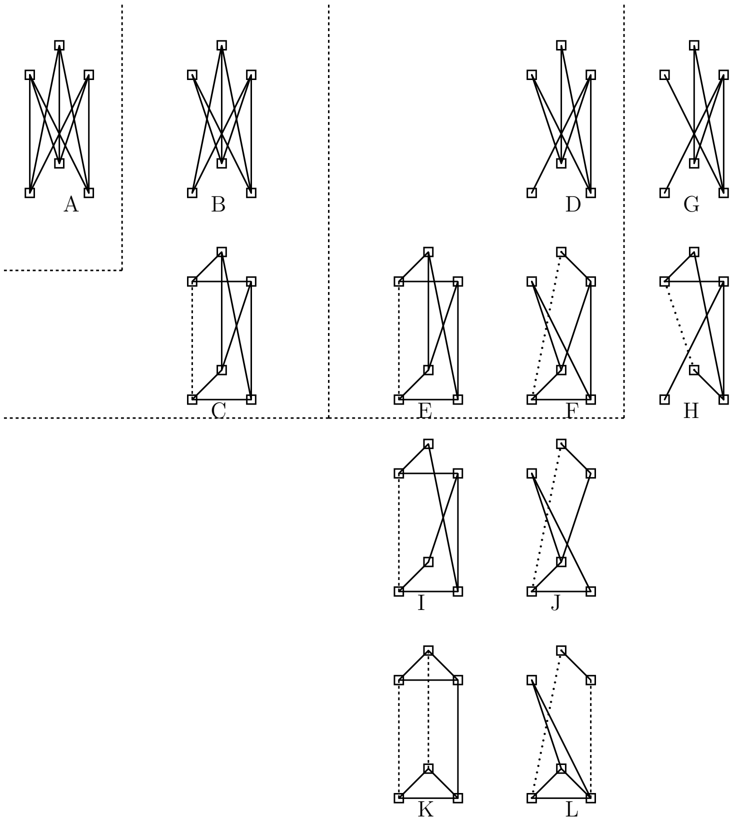

Figure 3. Decomposition of the complete graph via (41) (ignoring the top and bottom layer) into graphs with transfer matrix product form. The broken lines in the graphs represent *identifications* of nodes, as opposed to the solid lines which are the usual graph edges.

Proposition 19. *For* $\lambda \in \Gamma_Q$ *let* $S^Q_\lambda \cong \mathcal{L}_{\mathcal{I}^{-1}(\lambda)}$ *be an irreducible rep of* $\hat{P}_n(Q)$*. Then*

$$\mathrm{Res}^{\hat{P}_n}_{\hat{P}_{n-1}} S^Q_\lambda \cong \bigoplus_{\mu \triangleright\triangleleft \lambda} S^Q_\mu$$

(Note, the sum is over μ*s for which there exists a* μ' *such that* $\mu \triangleright \mu' \triangleleft \lambda$*, so e.g. the multiplicity of* S^Q_λ *on the right-hand side is the number of distinct values of* λ_i*.)*

Comparing with the Clebsch–Gordan calculation in (33) we recover (31) from this supposition. However, if any *more* rows or columns are lost then this would violate (31), thus the supposition is correct and the last two propositions are established. □

From proposition 19 the minimal spectrum multiplicities follow immediately (cf [24]). For example, below we give $\hat{M}_4$, the generating matrix of the $Q = 4$ case $\hat{P}_{n+1} \supset \hat{P}_n$ Bratteli diagram (cf alternate lines of figure 4) with its Perron eigenvector:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \\ 3 \\ 1 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 3 \\ 2 \\ 3 \\ 1 \end{pmatrix}. \tag{43}$$

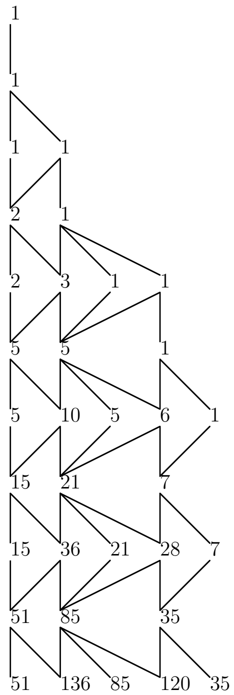

Figure 4. Bratteli diagram at $Q = 4$ for $\hat{P}_n \subset \hat{P}_n^1 \subset \hat{P}_{n+1}$ up to $n = 5$. From left to right the Γ_Q indices are $(4), (3,1), (2,2), (2,1^2), (1^4)$ and the Γ_{Q-1} indices are $(3), (2,1), (1^3)$. This should be compared with the conjunction of figure 2, proposition 9 of [34] and proposition 5 (via proposition 18). Alternate lines appear in [24].

Combining proposition 18 with proposition 13 it follows that for each non-negative integer $i \leqslant Q/2$, the propagating number i sector of $\mathcal{T}$ has a largest eigenvalue of multiplicity *exactly* $\dim S(\mathcal{I}((i)))$, for any set of ferromagnetic couplings.

6. Discussion

To summarize: we have shown how the partition algebra formalism matches up the correlation functions of the Potts model and the corresponding absolute spectrum degeneracies of its transfer matrix. Briefly, the multiplicity of the analytic function corresponding to m spin-to-m spin correlations with interchange symmetry characterized by partition $\lambda \vdash m$ is the dimension of the irreducible representation of S_Q indexed by partition $(Q-m, \lambda_1, \lambda_2, \ldots) \vdash Q$ if this *is* a partition (i.e. if $\lambda_1 \leqslant Q-m$) and is zero otherwise.

We see that the Potts model may be characterized by the $(Q-1, 1)$ Young diagram, in the sense that the maximal Potts algebra is the commutant of S_Q on tensored copies of the permutation representation with index $(Q-1, 1)$ [20]. This mirrors to some extent the relationship between $U_q sl_2$ and two-dimensional vertex models, and between $\mathrm{O}(Q)$ and the Brauer algebra [8,9,21]. Here the Hopf algebra (or crucially bi-algebra) is not a q-group but a group algebra, and the consequently greater symmetry of the form of the coproduct allows the building of higher-dimensional models (although so far without the Yang–Baxter equation—there may be a clue here for finding tetrahedron equations†). In a subsequent paper we will discuss the physicality of the models arising from tensoring other representations of groups, and from fusion!

That there is an S_Q symmetry of any classical Hamiltonian of the form of (2) when $h_i = 0$

† A 'ramified' form of the partition algebra has recently been proposed to pursue this point (joint work with Elgamal and Stanley on this topic is to be reported shortly).

is obvious. That it is the *only* symmetry of the corresponding transfer matrix is a phenomenon which, in two dimensions, is only seen in the special cases $Q = 2, 3$ [31]. Note that our S_Q/P_n duality implies that in order to locate the Q crossover point in three dimensions it is sufficient to locate a lowest Q value for which the absolute spectrum degeneracy differs from that determined by this duality *for any* n. It is reasonable to suppose that if such a Q-value exists the deviation should occur at a lattice size accessible by direct computation. This work is in progress.

Acknowledgments

I would like to thank the EPSRC (grant nos GRJ 25758 and GRJ 29923) and the Nuffield Foundation for financial support, and Tutul Dasmahapatra, Dave Woodcock, Genevieve Rollet, Jan Paradowski, Andrew Mathas and other members of the Donkin seminar for useful conversations. I would like to thank my colleagues at Kyoto University, and particularly Tetsuji Miwa, for a warm welcome there during the completion of this work, and to thank JSPS, RIMS Kyoto, the Royal Society and the Isaac Newton Institute for organizing and supporting this visit.

Appendix. Mathematical background

Appendix A.1. Categorical context and Young subalgebras for P_n

Here we discuss an analogue of the Young subgroup theory of S_n for partition algebras, in preparation for appendix A.2. For $\lambda = (\lambda_1, \lambda_2, \ldots)$ a dominant weight define a set

$$\overline{\lambda} = \bigcup_{i=1}^{\lambda'_1} \{\alpha^i\}_{\underline{\lambda_i}} \tag{44}$$

(so, for example, $\overline{(2,1)} = \{\alpha^1_1, \alpha^1_2, \alpha^2_1\}$). Let $b(\alpha^1, \alpha^2, \ldots) \in \boldsymbol{E}_{\overline{\lambda}}$ by a natural extension of our existing notation. We can make $k\boldsymbol{E}_{\overline{\lambda}}$ a P_n-module for $n = \lambda_i$ (any i) by defining a (left) multiplication

$$*^i_1 : P_{\lambda_i} \times k\boldsymbol{E}_{\overline{\lambda}} \to k\boldsymbol{E}_{\overline{\lambda}}$$

by

$$\begin{aligned} a(\alpha, \beta) *^i_1 b(\alpha^1, \alpha^2, \ldots, \alpha^{i-1}, \alpha^i, \alpha^{i+1}, \ldots) \\ = Q^{C_{\{\gamma\}_{\underline{n}}}(a(\alpha^i,\gamma)\bullet b(\alpha^1,\alpha^2,\ldots,\alpha^{i-1},\gamma,\alpha^{i+1},\ldots))} \\ \times R_{\{\gamma\}_{\underline{n}}}(a(\alpha^i, \gamma) \bullet b(\alpha^1, \alpha^2, \ldots, \alpha^{i-1}, \gamma, \alpha^{i+1}, \ldots)). \end{aligned} \tag{45}$$

Indeed for any M and $\lambda \vdash |M|$, any bijection

$$f : \overline{\lambda} \to M \tag{46}$$

defines a (left) multiplication

$$*^i_f : P_{\lambda_i} \times k\boldsymbol{E}_M \to k\boldsymbol{E}_M$$

for each i, which makes $k\boldsymbol{E}_M$ a P_{λ_i}-module. This multiplication is given by applying the bijection $\boldsymbol{E}^f$ to the $f = 1$ (identity map) version above. Further

$$a *^i_f (b *^j_f c) = b *^j_f (a *^i_f c) \qquad i \neq j$$

so that $k\boldsymbol{E}_M$ is a (left) $\bigotimes_i kP_{\lambda_i}(Q)$ module.

Conversely, we will see shortly that $\mathrm{End}_{\bigotimes_i kP_{\lambda_i}(Q)}(k\boldsymbol{E}_M) \cong \mathrm{Cen}_{\bigotimes_i kP_{\lambda_i}(Q)}(k\boldsymbol{E}_M)$.

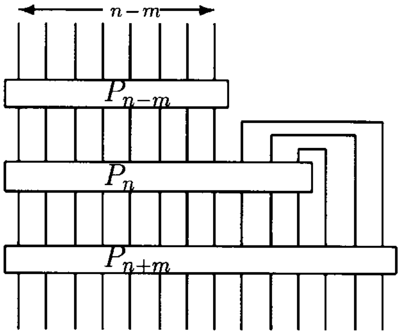

Figure 5. Schematic (cf figure 1) for the canonical action of $P_{n-m} \otimes P_{n+m}$ on P_n.

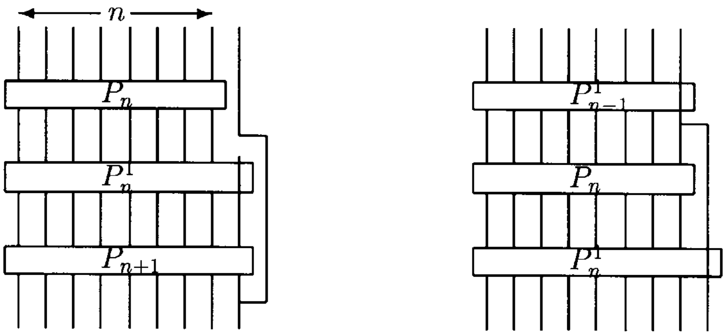

Figure 6. Action of $P_n \otimes P_{n+1}$ on P_n^1 (cf (45)), and of $P_{n-1}^1 \otimes P_n^1$ on P_n.

Recall that an associative algebra $\mathcal{A}$ is itself an $\mathcal{A}$ bimodule by the left and right multiplication in the algebra; and $\mathcal{A} \cong \mathrm{End}_{\mathcal{A}}(\mathcal{A})$. The multiplication $*^i_f$ allows us to interpret P_n as a left P_n module in many different ways. The basic 'left' and 'right' actions are canonical via algebra multiplication. Others may be defined by somehow specifying f.

Note from the definitions that, with $\mathcal{A}^{op}$ the opposite algebra of $\mathcal{A}$,

$$P_n \cong P_n^{op} \qquad P_n^1 \cong (P_n^1)^{op}$$

and hence via appropriate choices of λ in (46) we have (1) and (2) of the following proposition.

Proposition 20.

(1) For any $m \leqslant n$ then P_n is a (left) $P_{n-m} \otimes P_{n+m}$ module with the action indicated by figure 5.

(2) P_n^1 is a $P_n \otimes P_{n+1}$ module, with the action indicated in figure 6.

(3) P_n is a $P_{n-1}^1 \otimes P_n^1$ module, with the action indicated in figure 6.

In fact, P_n is obviously faithful as a P_{n-m} module (it has the regular module as a submodule by the inclusion $P_{n-m} \subset P_n$), but it is not a faithful P_{n+m} module. More precisely, let

$$e^n(m) = \left(\prod_{i=n-m+1}^{n+m} \frac{A^{i\cdot}}{Q} \right) \in P_{n+m}$$

then as a (left) $P_{n-m} \otimes P_{n+m}$ module $P_n \cong P_{n+m}\, e^n(m)$ (where the P_{n-m} action is obtained from the P_{n-m}^{op} action on the *right*). The isomorphism is given by the map in which the image

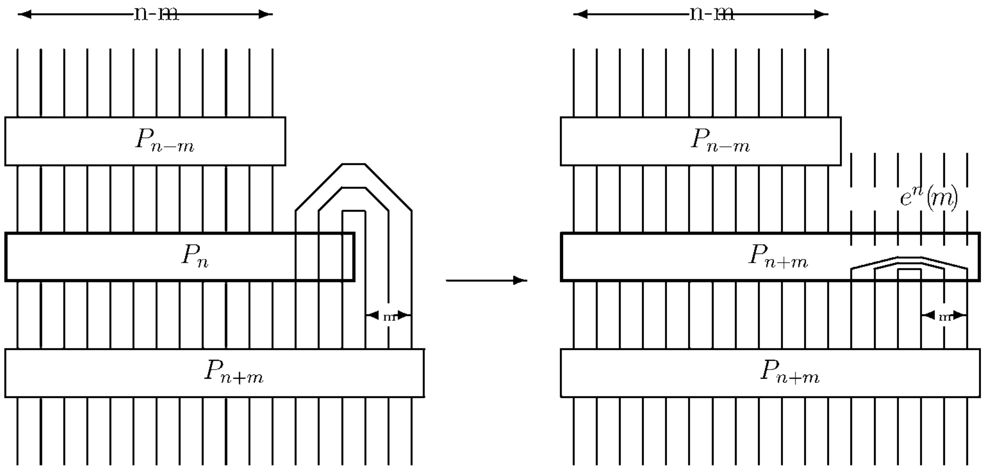

Figure 7. Schematic for the deformation isomorphism for (47).

of the P_n unit is

$$\begin{aligned}1_{P_n} = \{\{\alpha_1, \beta_1\}, \{\alpha_2, \beta_2\}, \ldots, \{\alpha_n, \beta_n\}\} \\ \mapsto \\ \{\{\alpha_1, \beta_1\}, \ldots, \{\alpha_{n-m}, \beta_{n-m}\}, \{\alpha_{n-m+1}\}, \ldots, \{\alpha_{n+m}\}, \\ \{\beta_{n-m+1}, \beta_{n+m}\}, \{\beta_{n-m+2}, \beta_{n+m-1}\}, \ldots, \{\beta_n, \beta_{n+1}\}\} \qquad (47)\end{aligned}$$

(consider deforming the diagram in figure 5 as shown in figure 7). Now $P_{n+m}\ e^n(m)$ is a (left) projective P_{n+m} module (since $e^n(m)$ is idempotent), and in general

$$\mathrm{Ann}_{P_{n+m}}(P_{n+m}\ e^n(m)) \neq 0.$$

Finally, we get part (1) of proposition 21.

Proposition 21.

(1)

$$P_{n+m}/\mathrm{Ann}_{P_{n+m}}(P_{n+m}\ e^n(m)) \cong \mathrm{End}_{P_{n-m}}(P_n).$$

(2) With respect to the $*^i_f$ *action in (45)*

$$\mathrm{End}_{\bigotimes_{i\neq j} kP_{\lambda_i}}(k\boldsymbol{E}_M) \cong \begin{cases} kP_{\lambda_j}/\mathrm{Ann}_{kP_{\lambda_j}}\left(kP_{\lambda_j}\left(\prod_{l=1}^{2\lambda_j-|M|} A^{l\cdot}\right)\right) & 2\lambda_j > |M| \\ kP_{\lambda_j} & 2\lambda_j = |M| \\ kP_{\lambda_j} + \mathrm{Cen}_{\bigotimes_{i\neq j} kP_{\lambda_i}}(k\boldsymbol{E}_M) & 2\lambda_j < |M| \end{cases}$$

(note the last sum is not necessarily direct).

Outline of the proof.

(1) Note that each number in the Bratteli diagram in figure 2 is the number of descending walks from the top of the diagram to that point. Thus, for example, the (leftmost) 'spine' dimension at level P_{2n} is also the dimension of the regular representation at level P_n, i.e. the sum of the squares of the given dimensions at that level. Extending this construction we have

$$|P_n| = \sum_{\lambda \in \Gamma^{n-m}} |\mathcal{S}_\lambda(n-m)| \times |\mathcal{S}_\lambda(n+m)|$$

where $|\mathcal{S}_\lambda(n-m)|$ denotes the dimension of the generically simple P_{n-m} module (for example $|P_3| = 203 = 2 \times 15 + 3 \times 37 + 1 \times 31 + 1 \times 31$) which proves the generic case. The general case then follows from the known structure of the regular representation of P_{n-m} [34].

(2) In the extreme case in which M is partitioned with shape (1^n) then all the simple modules are one-dimensional, and inequivalent ones are indexed by possible partitions of M (indeed, with an appropriate quotient each partition spans the corresponding simple), as required. The remaining cases follow as above. □

Appendix A.2. A Jones basic construction on $\hat{P}_*(Q)$

In this appendix we show that the $\hat{P}$ algebras satisfy the requirements for a Jones basic construction [15, 19].

Proposition 22. *For all n, Q the algebras $\hat{P}_n(Q)$ and $\hat{P}_n^1$ are semi-simple over $\mathbb{C}$.*

Proof. It is enough to show that for each non-zero $a \in \hat{P}$ there exists $b \in \hat{P}$ such that $\rho_n(ba)$ is non-zero Hermitian (since then there does not exist any nilpotent ideal). But in the faithful representation ρ_n the generators are all real symmetric matrices so $a^\dagger = a^{op*}$ (a^{op*} is a with generators written in reverse order and coefficients complex conjugated) and $a^\dagger a$ is non-zero Hermitian. □

Proposition 23. *The function*

$$\mathrm{tr}_n(-) = \mathrm{tr}(\rho_n(-))$$

is a non-degenerate trace on $\hat{P}_n(Q)$.

(That is to say, $\mathrm{tr}_n(ab) = \mathrm{tr}_n(ba)$ and for each $a \in \hat{P}_n(Q)$ there exists b such that $\mathrm{tr}_n(ba) \neq 0$).

Proof. From above note that $\mathrm{tr}_n(a^\dagger a) > 0$. □

Similarly for $\hat{P}_n^1$.

Note that $\mathrm{tr}_n(-)$ is also non-degenerate on $\hat{P}_{n-1} \subset \hat{P}_n$ since the restriction is

$$\mathrm{Res}_{\hat{P}_{n-1}}^{\hat{P}_n} \rho_n \cong \rho_{n-1} \oplus \rho_{n-1} \oplus \cdots \oplus \rho_{n-1} \qquad (Q \text{ copies})$$

and non-degenerate on $e\hat{P}_{n-1} \subset \hat{P}_n$ (with idempotent $e = \frac{1}{Q} A^{n\cdot}$) since

$$\rho(e) = 1_Q \otimes 1_Q \otimes \cdots \otimes 1_Q \otimes \frac{1}{Q} M$$

where the last factor may be similarity transformed to a diagonal matrix M' such that $(M')_{ii} = \delta_{i,1}$, whereupon the restriction is

$$\mathrm{Res}_{e\hat{P}_{n-1}}^{\hat{P}_n} \rho_n \cong M' \otimes \rho_{n-1}. \tag{48}$$

Definition 6 ('conditional expectation'). *Define*

$$\epsilon_{n-1} : \hat{P}_n \to \hat{P}_{n-1}$$

by

$$tr_n(ba) = \mathrm{tr}_n(\epsilon_{n-1}(b)a) \qquad \forall a \in \hat{P}_{n-1}.$$

Actually this is equivalent to the following construction. First define

$$\mathcal{E}_{n-1} : \hat{P}_n \to e\hat{P}_{n-1} \cong \hat{P}_{n-1}$$

by

$$\mathcal{E}_{n-1} : b \mapsto ebe = e\epsilon_{n-1}(b) \mapsto \epsilon_{n-1}(b)$$

then $a \in e\hat{P}_{n-1}$ may be written $ae = ea = eae$ and so $ba = beae$, and $\mathrm{tr}_n(ba) = \mathrm{tr}_n(beae) = \mathrm{tr}_n(ebea)$.

Thus

$$C = \mathrm{End}_{\hat{P}_{n-1}}(\hat{P}_n)$$

is a Jones basic construction [15, 19] (note here that $\hat{P}_n$ is a double sided ideal, but $\hat{P}_{n-1}$ acts only on the left).

Iterating from (37)

$$\mathrm{End}_{P_{n-m}}(P_n) \equiv P_{n+m} \qquad \mathrm{mod}\ \mathrm{Ann}_{P_{n+m}}(V_Q^{\otimes n+m})$$

(for $n - m > Q$) thus again from figure 5, putting $m = 1$ and using definition 4 and proposition 21 we see that $C \cong \hat{P}_{n+1}$, i.e. we have the following proposition.

Proposition 24.

$$\hat{P}_{n+1} \cong \mathrm{End}_{\hat{P}_{n-1}}(\hat{P}_n). \tag{49}$$

Note that the map $\mathcal{E}_n$ gives a conditional expectation on $\hat{P}_n^1$ to $\hat{P}_n$ (by restriction of the domain from $\hat{P}_{n+1}$) so that $D = \mathrm{End}_{\hat{P}_n}(\hat{P}_n^1)$ is another Jones basic construction, with $D \cong \hat{P}_{n+1}$.

Appendix A.3. Quasi-heredity

For completeness we show here the quasi-heredity property (as in Cline *et al* [10], Dlab and Ringel [13]) of $P_n^1(Q)$ explicitly.

Proposition 25 (heredity chain). *For $Q \neq 0$ the list $(E_{n+1}^{(n+1)}, E_n^{(n+1)}, \ldots, E_1^{(n+1)})$ is a heredity chain for $P_n^1(Q)$.*

Proof. Noting (15), we require (cf [10]) to show that $E_i^{(n+1)}$ is a heredity idempotent modulo $P_n^1 E_{i-1}^{(n+1)} P_n^1$.

This requires:

(i) that $E_i^{(n+1)} P_n^1 E_i^{(n+1)}$ is semi-simple mod $P_n^1 E_{i-1}^{(n+1)} P_n^1$;

(ii) that $P_n^1 E_i^{(n+1)} \otimes_{E_i^{(n+1)} P_n^1 E_i^{(n+1)}} E_i^{(n+1)} P_n^1 \to P_n^1 E_i^{(n+1)} P_n^1$ is bijective.

Now (i) follows on noting that $P_n^1 E_i^{(n+1)} P_n^1 / P_n^1 E_{i-1}^{(n+1)} P_n^1$ has basis $\{x \in \boldsymbol{E}_n^1 | \#^P(x) = i\}$ but the subset preserved (up to scalars) in $E_i^{(n+1)} P_n^1 E_i^{(n+1)}$ has the form

$$\{\{\alpha_1\}\{\beta_1\}\{\alpha_2\}\{\beta_2\} \ldots \{\alpha_{n-i+1}\}\{\beta_{n-i+1}\} \ldots\}$$

so that all remaining components must be paired so as to contribute positively to $\#^P$. Thus this quotient algebra is isomorphic to the group algebra over k of some permutation group on i objects (in fact it is the symmetric group S_{i-1} acting on the first $i - 1$ of these, and acts trivially on the last one—now recall that $\mathbb{C}S_{i-1}$ is semi-simple with simple modules indexed by partitions of the integer $i - 1$, denoted $\lambda \vdash i - 1$). Condition (ii) follows on noting that

$$aE_i^{(n+1)} \otimes E_i^{(n+1)} b \mapsto aE_i^{(n+1)} b$$

is obviously onto, and that injectivity may be proved similarly to the above permutation group argument, using the given basis. □

Definition 7. *Let $\lambda \vdash i$. Using Young-subgroup-style notation, $E_{\lambda;1} = E_0^{(n-i)} \otimes e_\lambda \otimes 1$ (case P_n^1 : the idempotent e_λ is as defined in (38), but acts in positions α_{n-i+1} through α_n); and $E_\lambda = E_0^{(n-i)} \otimes e_\lambda$ (case P_n).*

Proposition 26. *A* maximal heredity chain *for $P_n^1(Q)$ is obtained by replacing $E_{i+1}^{(n+1)}$ with $E_{\lambda^{(1)};1}, E_{\lambda^{(1)};1} + E_{\lambda^{(2)};1}, \ldots, \sum_{j=1}^{k} E_{\lambda^{(j)};1}, \ldots$, where $\lambda^{(j)} \vdash i$ and the list runs over all such (in any order).*

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